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The colorings of homeomorphisms on connected graphs

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ABSTRACT

In this paper, we decide the exact value of the color number of a fixed point free homeomorphism on a connected locally finite graph. We prove that for every fixed-point free homeomorphism from a connected locally finite graph into itself, the greatest common divisor of all period for its map is equal to one or three if and only if its color number is 4.

1. Introduction

All spaces are assumed separable metrizable and all maps are continuous. We denote the set of natural numbers by $\mathbb{N}$, and write the unit interval $[0,1]$ as $\mathbb{I}$. If $f: X \to X$ is a map, then we write inductively $f^0 = \mathrm{id}_X$ and $f^n = f \circ f^{n-1}$ for each $n \in \mathbb{N}$.

Let $f: X \to X$ be a fixed-point free map, i.e., $f(x) \neq x$ for all $x \in X$. A closed subset A of X is called a *color* of (X, f) if $f(A) \cap A = \emptyset$. A *coloring* of (X, f) is a cover $\mathcal{U}$ of X consisting of colors, i.e., $f(U) \cap U = \emptyset$ for all $U \in \mathcal{U}$. The minimal cardinality of a coloring is called the *color number* of (X, f), denoted by $\mathrm{col}(X, f)$. See [2]. Since finite open covers can be shrunk to closed covers, and finite closed covers can be swelled to open covers, the closedness of the coloring is irrelevant. Finite open covers do equally well.

Note that the color number may not be finite. For example, if f is the antipodal map on the unit sphere in Hilbert space, then its color number is infinite. For an upper bound of the color number, it was known the following.

Theorem 1.1. *([5, Theorem 3]) Let X be a paracompact Hausdorff space with* $\dim X \leqslant n$. *If $f: X \to X$ is a fixed-point free homeomorphism, then* $\mathrm{col}(X, f) \leqslant n + 3$.

In [8, Theorem 1.1], J. van Mill gives a simple proof of the theorem above. Furthermore, for a fixed-point free involution, the upper bound of the color number can be improved.

Theorem 1.2. *([3, Theorem 2]) Let X be a paracompact Hausdorff space with* $\dim X \leqslant n$ *and $f: X \to X$ a fixed-point free homeomorphism. If f is an involution, i.e., $f^2(x) = x$ for all $x \in X$, then* $\mathrm{col}(X, f) \leqslant n + 2$.

Let X be a connected space and $f: X \to X$ a fixed-point free homeomorphism. Clearly, $\mathrm{col}(X, f) \geqslant 3$. It was stated in [3, Example 7(1)] that if $f^3(x) = x$ for *all* $x \in X$, then $\mathrm{col}(X, f) \geqslant 4$. See [1] for the proof. In [1], it is proved that if X is an arcwise-connected space, if there exists $n \in \mathbb{N}$ such that $f^n = \mathrm{id}_X$, and if f has a point of period 3, then $\mathrm{col}(X, f) \geqslant 4$.

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From Theorem 1.1, it is of interest to know whether $\mathrm{col}(X, f) = n+3$ or $\mathrm{col}(X, f) \leqslant n+2$. In particular, we are interested in the following question. See Question 1.2 in [6].

Question 1.3. Let X be a 1-dimensional connected space and $f : X \to X$ a fixed-point free homeomorphism from X into itself. Which is true, $\mathrm{col}(X, f) = 3$ or $\mathrm{col}(X, f) = 4$?

The following question appear in [1] (see [7] for graphs).

Question 1.4. Let X be a connected finite graph and $f : X \to X$ a fixed-point free homeomorphism on X. Which is true, $\mathrm{col}(X, f) = 3$ or $\mathrm{col}(X, f) = 4$?

Let $x \in X$ and $f : X \to X$ a map. We call $\mathrm{Orb}(x, f) = \{f^n(x)\colon\ n \geqslant 0\}$ the *orbit of x for f*. An orbit $\mathrm{Orb}(x, f)$ is said to be a *p-periodic orbit of x for f* if $x = f^p(x)$, where let $p = |\mathrm{Orb}(x, f)|$. Set $\mathrm{Per}(f) = \{|\mathrm{Orb}(x, f)|\colon\ x \in X$ and $\mathrm{Orb}(x, f)$ is a periodic orbit of x for $f\}$. Let A be a subset of $\mathbb{N}$. We write the greatest common divisor of A as $\gcd(A)$ or $\gcd(p_0, \dots, p_k)$ if $A = \{p_0, \dots, p_k\}$. Let $f : X \to X$ be a fixed-point free homeomorphism from a connected finite graph X into itself with $\mathrm{Per}(f) \neq \emptyset$. In [1] it is shown that if $\gcd(\mathrm{Per}(f)) \notin \{1, 3\}$, then $\mathrm{col}(X, f) = 3$. The aim of this paper is to prove the following which is the complete answer to Question 1.4.

Theorem 1.5. *Let $f : X \to X$ be a fixed-point free homeomorphism from a connected locally finite graph X to itself. Then* $\gcd(\mathrm{Per}(f)) \in \{1, 3\}$ *if and only if* $\mathrm{col}(X, f) = 4$.

By [1], to prove Theorem 1.5, it suffices to show the following:

(1) If $\gcd(\mathrm{Per}(f)) \in \{1, 3\}$, then $\mathrm{col}(X, f) = 4$ (in fact, it is shown that $\gcd(\mathrm{Per}(f)) \in \{1, 3\}$ and $\mathrm{col}(X, f) = 3$ is impossible) (see Theorem 5.3 below).
(2) If $\mathrm{Per}(f) = \emptyset$, then $\mathrm{col}(X, f) = 3$ (see Theorem 5.9 below).
(3) If X is an infinite graph and if $\mathrm{Per}(f)$ is nonempty satisfying $\gcd(\mathrm{Per}(f)) \notin \{1, 3\}$, then $\mathrm{col}(X, f) = 3$ (see Theorem 5.11 below).

Most of this paper is the proof of (1) above. Thus, we examine 3-colorings of a fixed-point free homeomorphism f from a connected locally finite graph X into itself and introduce an equivalence relation on the set $\mathfrak{C}_p$ of all 3-colorings of the shift map $s : \mathbb{Z}_p \to \mathbb{Z}_p$ in Section 2. By the definition of this relation, it is easily seen that if f has a 3-periodic point, then $\mathrm{col}(X, f) = 4$. In Section 3, we give a quantity of the relation above. And, in Section 4, we examine a relation between this quantities of two finite orbits of f and determines the value of this quantity on $\mathfrak{C}_p$. The results above drive a contradiction. In Section 5, some applications and a counterexample are indicated.

2. Coloring for $(\mathbb{Z}_p, s)$

In this section, it is shown that if a fixed-point free homeomorphism f from a connected locally finite graph X to itself has a 3-periodic point, then $\mathrm{col}(X, f) = 4$. So, we set up notation and definition.

Notation 2.1. Let $p \in \mathbb{N}$ with $p \geqslant 2$ and $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ a p-points discrete space. Let $s_p : \mathbb{Z}_p \to \mathbb{Z}_p$ be a homeomorphism defined by $s_p(i) = i + 1$ for each $i = 0, \dots, p-1$, where we identify p with 0. For simplicity of notation, we write s instead of s_p.

Notation 2.2. Let $\mathfrak{C}_p$ be the set of all 3-colorings $\mathcal{U} = \{U_1, U_2, U_3\}$ of $(\mathbb{Z}_p, s)$ satisfying that $U_i \cap U_j = \emptyset$ whenever $i \neq j$.
Let $\mathcal{U} = \{U_1, U_2, U_3\} \in \mathfrak{C}_p$ such that $i \in U_{k_i}$ for some $k_i \in \{1, 2, 3\}$. We can write $\mathcal{U}$ as

$$k_0 k_1 \dots k_{p-1}.$$

For example, let $U_1 = \{1, 3\}$, $U_2 = \{0, 2\}$, and, $U_3 = \{4\}$. Then $\mathcal{U} = \{U_1, U_2, U_3\} \in \mathfrak{C}_5$. By the above, we write $\mathcal{U}$ as

$$21213.$$

First, we introduce an equivalence relation on $\mathfrak{C}_p$.

Definition 2.3. Let $\mathcal{U} = k_0 \dots k_{p-1} \in \mathfrak{C}_p$ as in Notation 2.2 and $i \in \{0, \dots, p-1\}$ such that $k_{i-1} = k_{i+1}$. We note $k_i \neq k_{i+1}$. Set $\{\ell_i\} = \{1, 2, 3\} \setminus \{k_i, k_{i+1}\}$ and

$$\mathcal{V} = k_0 \dots k_{i-1} \ell_i k_{i+1} \dots k_{p-1}.$$

We note that $\mathcal{V} \in \mathfrak{C}_p$. For simplicity of notation, we write $\mathcal{U} \sim_i \mathcal{V}$. For example, if let $\mathcal{U} = 1213212$, $\mathcal{V} = 1213232 \in \mathfrak{C}_7$, then $\mathcal{U} \sim_5 \mathcal{V}$ (or $1213212 \sim_5 1213232$).

If $\mathcal{U}_0 \sim_{i_1} \mathcal{U}_1 \sim_{i_2} \cdots \sim_{i_n} \mathcal{U}_n$, we write $\mathcal{U}_0 \sim \mathcal{U}_n$ or $\mathcal{U}_0 \sim_A \mathcal{U}_n$, where A satisfies $\{i_1, \dots, i_n\} \subset A \subset \mathbb{Z}_p$. We note that $\sim$ is an equivalence relation on $\mathfrak{C}_p$.

Notation 2.4. Let $f : X \to X$ be a fixed-point free homeomorphism from a space X to itself, $\mathcal{U}$ a coloring for (X, f), and, Y a subspace of X with $f(Y) = Y$. Write $\mathcal{U}|_Y = \{U \cap Y\colon\ U \in \mathcal{U}\}$ which is a coloring for $(Y, f|_Y)$.

Definition 2.5. Let $f : X \to X$ be a fixed-point free homeomorphism from a space X to itself, $x \in X$ such that $\mathrm{Orb}(x, f)$ is a p-periodic orbit of x for f, and, $\mathcal{U}$ a coloring for (X, f). Then, there exists a homeomorphism $h : \mathrm{Orb}(x, f) \to \mathbb{Z}_p$ such that $h(x) = 0$ and $h \circ f|_{\mathrm{Orb}(x,f)} = s_p \circ h$.

$$\begin{tikzcd}
\mathrm{Orb}(x,f) \arrow[r, "f|_{\mathrm{Orb}(x,f)}"] \arrow[d, "h"'] & \mathrm{Orb}(x,f) \arrow[d, "h"] \\
\mathbb{Z}_p \arrow[r, "s_p"'] & \mathbb{Z}_p
\end{tikzcd}$$

Thus, we can identify $(\mathrm{Orb}(x, f), f|_{\mathrm{Orb}(x,f)}, \mathcal{U}|_{\mathrm{Orb}(x,f)})$ with $(\mathbb{Z}_p, s_p, h(\mathcal{U}|_{\mathrm{Orb}(x,f)}))$, where $h(\mathcal{U}|_{\mathrm{Orb}(x,f)}) = \{h(U \cap \mathrm{Orb}(x, f))\colon\ U \in \mathcal{U}\}$ is a coloring for $(\mathbb{Z}_p, s_p)$.

Definition 2.6. Let $f : X \to X$ be a fixed-point free homeomorphism and $\mathcal{U}$ a coloring of (X, f). A sequence of points, $\{x_n\colon\ 0 \leqslant n \leqslant p-1\}$, is said to be a *$p$-periodic $\mathcal{U}$-pseudo orbit of f* if for every $0 \leqslant n \leqslant p-1$ there exist $U_n \in \mathcal{U}$ and a connected subset A_n of U_n containing $\{x_{n+1}, f(x_n)\}$ such that $U_n \neq U_{n+1}$, where let $x_p = x_0$ and $U_p = U_0$.

Theorem 2.7. *Let $f : \mathbb{Z}_p \times \mathbb{I} \to \mathbb{Z}_p \times \mathbb{I}$ be a homeomorphism such that $f(i, k) = (s(i), k)$ for each $i \in \mathbb{Z}_p$ and $k = 0, 1$, and, $\mathcal{U} = \{U_1, U_2, U_3\}$ a 3-coloring of $(\mathbb{Z}_p \times \mathbb{I}, f)$ such that $\mathcal{U}|_{\mathrm{Orb}(x_k,f)} \in \mathfrak{C}_p$ for $k = 0, 1$, where $x_k = (0, k)$. Then $\mathcal{U}|_{\mathrm{Orb}(x_0,f)} \sim \mathcal{U}|_{\mathrm{Orb}(x_1,f)}$.*

Proof. We may assume that $\mathcal{U}|_{\mathrm{Orb}(x_0,f)} \neq \mathcal{U}|_{\mathrm{Orb}(x_1,f)}$, $\mathrm{Cl}_{\mathbb{Z}_p \times \mathbb{I}}(\mathrm{Int}_{\mathbb{Z}_p \times \mathbb{I}} U_j) = U_j$ for each $j = 1, 2, 3$, and, T is finite with $T \cap (\mathbb{Z}_p \times \{0, 1\}) = \emptyset$, where $T = \bigcup_{j \neq j'} U_j \cap U_{j'}$. Moreover we may assume that $z \neq f^i(z')$ for any $z, z' \in T$ with $z \neq z'$ and each $i \in \{0, \dots, p-1\}$.

If $(i, t), (i, s) \in \mathbb{Z}_p \times \mathbb{I}$ with $t \leqslant s$, write $(i, t) \leqslant (i, s)$. Set $x_{0,i} = (i, 0) \in \mathbb{Z}_p \times \mathbb{I}$ for $0 \leqslant i \leqslant p-1$. Denote $\mathcal{U}|_{\mathbb{Z}_p \times \{0\}}$ by $\mathcal{U}_0 = k_{0,0} \dots k_{0,p-1} \in \mathfrak{C}_p$, where let $k_{0,i} \in \{1, 2, 3\}$ for each $i \in \{0, \dots, p-1\}$. There exist $i_1 \in \{0, \dots, p-1\}$, a p-periodic $\mathcal{U}$-pseudo orbit $\{x_{1,i}\colon\ 0 \leqslant i \leqslant p-1\}$ of f, and, a connected subset $C_{1,i}$ of $U_{k_{0,i}}$ connecting between $x_{0,i}$ and $x_{1,i}$ $(i \in \{0, \dots, p-1\})$ such that $x_{0,i} \leqslant x_{1,i}$ for each $i = 0, \dots, p-1$ and $x_{1,i_1} \in T$. Thus, we have $\{k_{1,i_1}\} = \{1, 2, 3\} \setminus \{k_{0,i_1}, k_{0,i_1+1}\}$. Set $k_{1,i} = k_{0,i}$ if $i \neq i_1$ and $\mathcal{U}_1 = k_{1,0} \dots k_{1,p-1} \in \mathfrak{C}_p$. It is clear that $\mathcal{U}_0 \sim_{i_1} \mathcal{U}_1$. Set $n_0 = |T|$. Inductively, for every $n \in \{2, \dots, n_0 + 1\}$, there exist $i_n \in \{0, \dots, p-1\}$, a p-periodic $\mathcal{U}$-pseudo orbit $\{x_{n,i}\colon\ 0 \leqslant i \leqslant p-1\}$ of f, and, a connected subset $C_{n,i}$ of $U_{k_{n,i}}$ connecting between $x_{n-1,i}$ and $x_{n,i}$ $(i \in \{0, \dots, p-1\})$ such that $x_{n-1,i} \leqslant x_{n,i}$ for each $i = 0, \dots, p-1$, $x_{n,i_n} \in T$, and $x_{n_0+1,i} = (i, 1) \in \mathbb{Z}_p \times \mathbb{I}$ for $0 \leqslant i \leqslant p-1$. Moreover, we have $\{k_{n,i_n}\} = \{1, 2, 3\} \setminus \{k_{n-1,i_n}, k_{n-1,i_n+1}\}$. Set $k_{n,i_n} = k_{n-1,i}$ if $i \neq i_n$ and $\mathcal{U}_n = k_{n,0} \dots k_{n,p-1} \in \mathfrak{C}_p$. It is clear that $\mathcal{U}_{n-1} \sim_{i_n} \mathcal{U}_n$ for each $n \in \{1, \dots, n_0 + 1\}$. Hence, $\mathcal{U}|_{\mathrm{Orb}(x_0,f)} = \mathcal{U}_0 \sim \mathcal{U}_{n_0+1} = \mathcal{U}|_{\mathrm{Orb}(x_1,f)}$. $\square$

Example 2.8. Let $p, q \in \mathbb{N}$ with $p, q \geqslant 2$, $\mathbb{Z}_p * \mathbb{Z}_q$ a join of $\mathbb{Z}_p = \{z_{p,0}, \dots, z_{p,p-1}\}$ and $\mathbb{Z}_q = \{z_{q,0}, \dots, z_{q,q-1}\}$, and $s_{p,q} = s_p * s_q : \mathbb{Z}_p * \mathbb{Z}_q \to \mathbb{Z}_p * \mathbb{Z}_q$ the natural fixed-point free homeomorphism satisfying that $s_{p,q}(z_{p,i} * z_{q,j}) = s_p(z_{p,i}) * s_q(z_{q,j})$ for each $i = 0, \dots, p-1$ and each $j = 0, \dots, q-1$.

Let $\mathcal{U} = \{U_1, U_2, U_3\}$ be a 3-coloring of $(\mathbb{Z}_p * \mathbb{Z}_q, s_{p,q})$. By shrinking and swelling $\mathcal{U}$, we may assume that $T = \bigcup_{j \neq j'} U_j \cap U_{j'}$ is finite, $\mathcal{U}|_{\mathrm{Orb}(z_{p,0}, s_{p,q})} \in \mathfrak{C}_p$ and $\mathcal{U}|_{\mathrm{Orb}(z_{q,0}, s_{p,q})} \in \mathfrak{C}_q$. If $p = q$, by Theorem 2.7, we have $\mathcal{U}|_{\mathrm{Orb}(z_{p,0}, s_{p,q})} \sim \mathcal{U}|_{\mathrm{Orb}(z_{q,0}, s_{p,q})}$. Suppose that $\gcd(p, q) = 1$. Set $r = p \times q$. Clearly, there exist a map $\phi : \mathbb{Z}_r \times \mathbb{I} \to \mathbb{Z}_p * \mathbb{Z}_q$ and a homeomorphism $f : \mathbb{Z}_r \times \mathbb{I} \to \mathbb{Z}_r \times \mathbb{I}$ as in Theorem 2.7 such that $\phi \circ f = s_{p,q} \circ \phi$.

$$\begin{tikzcd}
\mathbb{Z}_r \times \mathbb{I} \arrow[r, "f"] \arrow[d, "\phi"'] & \mathbb{Z}_r \times \mathbb{I} \arrow[d, "\phi"] \\
\mathbb{Z}_p * \mathbb{Z}_q \arrow[r, "s_{p,q}"'] & \mathbb{Z}_p * \mathbb{Z}_q
\end{tikzcd}$$

Set $\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U_1), \phi^{-1}(U_2), \phi^{-1}(U_3)\}$ and $x_k = (0, k) \in \mathbb{Z}_r \times \mathbb{I}$ for $k = 0, 1$. By Theorem 2.7, $\phi^{-1}(\mathcal{U})|_{\mathrm{Orb}(x_0,f)} \sim \phi^{-1}(\mathcal{U})|_{\mathrm{Orb}(x_1,f)}$.

The example above gives the ideas of the proof of the following lemma.

Lemma 2.9. *Let $f : X \to X$ be a fixed-point free homeomorphism from a connected locally finite graph X to itself, let $\mathrm{Orb}(x_k, f)$ be a p_k-periodic orbit of x_k for f, let A be an arc connecting between x_0 and x_1 in X, and, let $\mathcal{U} = \{U_1, U_2, U_3\}$ be a 3-coloring for (X, f) with $\mathcal{U}|_{\mathrm{Orb}(x_k, f)} \in \mathfrak{C}_{p_k}$ for $k = 0, 1$. Then there exist $p \in \mathbb{N}$, a map $\phi : \mathbb{Z}_p \times \mathbb{I} \to Z$ and a homeomorphism $\widetilde{f} : \mathbb{Z}_p \times \mathbb{I} \to \mathbb{Z}_p \times \mathbb{I}$ as in Theorem 2.7 such that $\phi(x'_k) = x_k$ for $k = 0, 1$, $\phi \circ \widetilde{f} = f|_Z \circ \phi$, and, $\phi^{-1}(\mathcal{U}|_Z)|_{\mathrm{Orb}(x'_0, \widetilde{f})} \sim \phi^{-1}(\mathcal{U}|_Z)|_{\mathrm{Orb}(x'_1, \widetilde{f})}$, where $x'_k = (0, k) \in \mathbb{Z}_p \times \mathbb{I}$ for $k = 0, 1$ and $Z = \bigcup_{i=0}^{p-1} f^i(A)$ satisfies $f(Z) = Z$.*

Proof. It is easily seen from Theorem 2.7 and the existence of $p = \min\{m \in \mathbb{N}\colon\ f^m(A) = A,\ f^m(x_k) = x_k \text{ for } k = 0, 1\}$. $\square$

Theorem 2.10. *Let $f : X \to X$ be a fixed-point free homeomorphism from a connected locally finite graph X to itself with 3-periodic point. Then $\mathrm{col}(X, f) = 4$.*

Proof. On the contrary, suppose that there exists a 3-coloring $\mathcal{U} = \{U_1, U_2, U_3\}$ for (X, f). Let $x \in X$ be a 3-periodic point for f. Without loss of generality we can assume that $x \in U_1$, $f(x) \in U_2$, $f^2(x) \in U_3$, and, that $\mathcal{U}|_{\mathrm{Orb}(x, f)}, \mathcal{U}|_{\mathrm{Orb}(f(x), f)} \in \mathfrak{C}_3$. Since X is connected, there exist an arc A connecting between x and $f(x)$, $p \in \mathbb{N}$, a map $\phi : \mathbb{Z}_p \times \mathbb{I} \to Z$ and a homeomorphism $\widetilde{f} : \mathbb{Z}_p \times \mathbb{I} \to \mathbb{Z}_p \times \mathbb{I}$ as in Theorem 2.7. Set $x'_k = (0, k) \in \mathbb{Z}_p \times \mathbb{I}$ for $k = 0, 1$. By Definition 2.3, we can write $\phi^{-1}(\mathcal{U}|_Z)|_{\mathrm{Orb}(x'_0, \widetilde{f})} = 123123\ldots123$ and $\phi^{-1}(\mathcal{U}|_Z)|_{\mathrm{Orb}(x'_1, \widetilde{f})} = 231231\ldots231$. It is clear from Definition 2.3 that $\phi^{-1}(\mathcal{U}|_Z)|_{\mathrm{Orb}(x'_0, \widetilde{f})} \nsim \phi^{-1}(\mathcal{U}|_Z)|_{\mathrm{Orb}(x'_1, \widetilde{f})}$, contrary to Lemma 2.9. $\square$

3. A quantity of the equivalence relation in Section 2

Notation 3.1. Let $\mathcal{U} = k_0 k_1 \ldots k_{p-1} \in \mathfrak{C}_p$. Set $\mathrm{EC}(\mathcal{U}) = \{i \in \mathbb{Z}_p\colon\ \mathcal{V} \in \mathfrak{C}_p,\ \mathcal{U} \sim_i \mathcal{V}\}$. If $i \in \mathrm{EC}(\mathcal{U})$, for simplicity of notation, we can write $\mathcal{U}$ as

$$k_0 k_1 \ldots \check{k}_i \ldots k_{p-1}.$$

For example, if $\mathcal{U} = 13212 \in \mathfrak{C}_5$, we have $\mathrm{EC}(\mathcal{U}) = \{3, 4\}$ and we can write $\mathcal{U}$ as

$$132\check{1}\check{2}.$$

Notation 3.2. Let $\mathcal{U} \in \mathfrak{C}_p$. Denote the equivalence class of $\sim$ containing $\mathcal{U} \in \mathfrak{C}_p$ by $[\mathcal{U}] \in \mathfrak{C}_p/\sim$. Set $\ell([\mathcal{U}]) = \max\{|\mathrm{EC}(\mathcal{V})|\colon\ \mathcal{U} \sim \mathcal{V}\}$.

Example 3.3.

(1) For every $\mathcal{W} \in \mathfrak{C}_3$, we have $\mathrm{EC}(\mathcal{W}) = \emptyset$, thus, $\ell([\mathcal{W}]) = 0$.
(2) Let $\mathcal{U} = 1\check{3}123\check{1}32 \in \mathfrak{C}_8$ as in Notation 3.1. It is clear that $\mathrm{EC}(\mathcal{U}) = \{1, 5\}$. Since

$$1\check{3}123\check{1}32 \sim_1 \check{1}\check{2}\check{1}23\check{1}32 \sim_2 \check{1}2\check{3}\check{2}3\check{1}32 \sim_0 \check{3}\check{2}\check{3}\check{2}3\check{1}3\check{2} \sim_5 \check{3}\check{2}\check{3}\check{2}\check{3}\check{2}\check{3}\check{2},$$

we have $\ell([\mathcal{U}]) = 8$.

From Lemma 2.9, $\ell([\phi^{-1}(\mathcal{U}|_Z)|_{\mathrm{Orb}(x'_0, \widetilde{f})}]) = \ell([\phi^{-1}(\mathcal{U}|_Z)|_{\mathrm{Orb}(x'_1, \widetilde{f})}])$. Section 4 establishes the relation between $\ell([\phi^{-1}(\mathcal{U}|_Z)|_{\mathrm{Orb}(x'_k, \widetilde{f})}])$ and $\ell([\mathcal{U}|_{\mathrm{Orb}(x_k, f)}])$ and determines $\{\ell([\mathcal{V}])\colon\ \mathcal{V} \in \mathfrak{C}_p\}$. And, when $\gcd(\mathrm{Per}(f)) \in \{1, 3\}$ and $\mathrm{col}(X, f) = 3$, these results introduce a contradiction in Section 5. In this section, we present some preliminaries. By Definition 2.3, it is easy to check the following.

Lemma 3.4. *Let $\mathcal{U} = k_0 \ldots k_{i-1} k_i k_{i+1} \ldots k_{p-1}$, $\mathcal{V} = k_0 \ldots k_{i-1} \ell_i k_{i+1} \ldots k_{p-1} \in \mathfrak{C}_p$ with $k_i \neq \ell_i$. Then, we can write $\mathcal{U} \sim_i \mathcal{V}$ as one of the following*:

(1) $\ldots k_{i-1} \check{k}_i k_{i+1} \ldots \sim_i \ldots \check{k}_{i-1} \check{\ell}_i \check{k}_{i+1} \ldots$ *or* $\ldots \check{k}_{i-1} \check{k}_i \check{k}_{i+1} \ldots \sim_i \ldots k_{i-1} \check{\ell}_i k_{i+1} \ldots$.
(2) $\ldots k_{i-1} \check{k}_i \check{k}_{i+1} \ldots \sim_i \ldots \check{k}_{i-1} \check{\ell}_i k_{i+1} \ldots$ *or* $\ldots \check{k}_{i-1} \check{k}_i k_{i+1} \ldots \sim_i \ldots k_{i-1} \check{\ell}_i \check{k}_{i+1} \ldots$.

Lemma 3.5. *Let $\mathcal{U} \in \mathfrak{C}_p$. Then $|\mathrm{EC}(\mathcal{U})| \neq p - 1$. Moreover, if p is odd, then $|\mathrm{EC}(\mathcal{U})| \neq p - 1, p$.*

Proof. Suppose that $|\mathrm{EC}(\mathcal{U})| = p - 1$ for some $\mathcal{U} = \{U_1, U_2, U_3\} \in \mathfrak{C}_p$. We may assume that $p - 1 \notin \mathrm{EC}(\mathcal{U})$, $0 \in U_1$, and, $1 \in U_2$. Since $0 \in \mathrm{EC}(\mathcal{U})$, we note that $p - 1 \in U_2$. Since $1 \in \mathrm{EC}(\mathcal{U})$, we have $2 \in U_1$. We continue in this fashion obtaining that $i \in U_1$ if $i \leqslant p - 2$ is even and $i \in U_2$ if $i \leqslant p - 2$ is odd. If $p - 2$ is even, then $p - 2 \in U_1$, thus, $p - 1 \in \mathrm{EC}(\mathcal{U})$, a contradiction. If $p - 2$ is odd, then $p - 2 \in U_2 \cap \mathrm{EC}(\mathcal{U})$ and $p - 1 \in U_1$, a contradiction. When p is odd, similarly, we can show that $|\mathrm{EC}(\mathcal{U})| \neq p - 1, p$. $\square$

Example 3.6.

(1) Let $U_1 = \{1,3,5,7\}$, $U_2 = \{0,2,4,6\}$, and, $U_3 = \{8\}$. Then $\mathcal{U} = \{U_1, U_2, U_3\} \in \mathfrak{C}_9$. By Notation 3.1, we write $\mathcal{U}$ as

$$2\check{1}\check{2}\check{1}\check{2}\check{1}\check{2}13.$$

We have two sequences as follows

$$2\check{1}\check{2}\check{1}\check{2}\check{1}\check{2}13 \sim_1 \check{2}\check{3}2\check{1}\check{2}\check{1}\check{2}13 \sim_3 \check{2}\check{3}\check{2}\check{3}2\check{1}\check{2}13 \sim_5 \check{2}\check{3}\check{2}\check{3}\check{2}\check{3}213,$$

and

$$2\check{1}\check{2}\check{1}\check{2}\check{1}\check{2}13 \sim_6 2\check{1}\check{2}\check{1}\check{2}1\check{3}\check{1}3 \sim_4 2\check{1}\check{2}1\check{3}\check{1}\check{3}\check{1}3 \sim_2 21\check{3}\check{1}\check{3}\check{1}\check{3}\check{1}3.$$

(2) Let $\mathcal{U} = 1\check{3}12\check{3}\check{2}\check{3}2 \in \mathfrak{C}_8$ as in Notation 3.1. We have a sequence as follows

$$1\check{3}12\check{3}\check{2}\check{3}2 \sim_5 1\check{3}1\check{2}\check{1}2\check{3}2 \sim_7 1\check{3}1\check{2}\check{1}\check{2}\check{1}\check{2}.$$

By Lemma 3.4, we can show the following from similar arguments in Example 3.6.

Lemma 3.7. *Let $\mathcal{U} \in \mathfrak{C}_p$ and $Z = \{s, s+1, \dots, s+t\} \subset \mathrm{EC}(\mathcal{U})$ such that $Z + \delta \not\subset \mathrm{EC}(\mathcal{U})$ for $\delta = -1, 1$, where $Z \pm 1 = \{i \pm 1\colon\ i \in Z\}$.*

(1) *If $|Z|$ is even, then there exist $\mathcal{L}, \mathcal{R} \in \mathfrak{C}_p$ such that $\mathcal{U} \sim_Z \mathcal{L}$, $\mathcal{U} \sim_Z \mathcal{R}$, $Z - 1 \subset \mathrm{EC}(\mathcal{L}) \not\supset Z$, $Z \not\subset \mathrm{EC}(\mathcal{R}) \supset Z + 1$, and, $|\mathrm{EC}(\mathcal{U})| = |\mathrm{EC}(\mathcal{L})| = |\mathrm{EC}(\mathcal{R})|$.*
(2) *If $|Z|$ is odd, then there exists $\mathcal{E} \in \mathfrak{C}_p$ such that $\mathcal{U} \sim_Z \mathcal{E}$, $(Z-1) \cup Z \cup (Z+1) \subset \mathrm{EC}(\mathcal{E})$, and, $|\mathrm{EC}(\mathcal{E})| = |\mathrm{EC}(\mathcal{U})| + 2$.*

It is easily seen from Lemmas 3.5 and 3.7 that $|\mathrm{EC}(\mathcal{U})|$ is even for each $\mathcal{U} \in \mathfrak{C}_p$.

4. Extensions

Notation 4.1. Let $\mathcal{U} \in \mathfrak{C}_p$. Set $\mathrm{O}(\mathcal{U}) = \{Z = \{s, s+1, \dots, s+t\} \subset \mathrm{EC}(\mathcal{U})\colon\ Z + \delta \not\subset \mathrm{EC}(\mathcal{U})$ for $\delta = -1, 1$, $|Z|$ is odd$\}$.

Let $\mathcal{U}, \mathcal{V} \in \mathfrak{C}_p$ such that $\mathcal{U} \sim \mathcal{V}$ and $\mathrm{O}(\mathcal{V}) = \emptyset$. First, this section proves $\ell([\mathcal{U}]) = |\mathrm{EC}(\mathcal{V})|$. By this result, we establishes the relation between $\ell([\phi^{-1}(\mathcal{U}|_Z)|_{\mathrm{Orb}(x'_k, \widetilde{f})}])$ and $\ell([\mathcal{U}|_{\mathrm{Orb}(x_k, f)}])$ in Lemma 2.9 and determines $\{\ell([\mathcal{V}])\colon\ \mathcal{V} \in \mathfrak{C}_p\}$. So, we introduce a following definition.

Definition 4.2. Let $\mathcal{U} \in \mathfrak{C}_p$. By Lemma 3.7(2), for every $Z = \{s, s+1, \dots, s+t\} \in \mathrm{O}(\mathcal{U})$, there exists $\mathcal{U}_1 \in \mathfrak{C}_p$ such that $\mathcal{U} \sim_Z \mathcal{U}_1$ and $(Z-1) \cup Z \cup (Z+1) \subset \mathrm{EC}(\mathcal{U}_1)$. We call $\mathcal{U}_1$ an *elementary extension of* $\mathcal{U}$, write $\mathcal{U} \to_s \mathcal{U}_1$ or $\mathcal{U} \to \mathcal{U}_1$. By Lemma 3.7(2), $|\mathrm{EC}(\mathcal{U}_1)| = |\mathrm{EC}(\mathcal{U})| + 2$. Thus, for every $\mathcal{U} \in \mathfrak{C}_p$ we have a sequence $\mathcal{U}_1, \dots, \mathcal{U}_n, \widetilde{\mathcal{U}}$ of $\mathfrak{C}_p$ such that $\mathcal{U} \to \mathcal{U}_1 \to \cdots \to \mathcal{U}_n \to \widetilde{\mathcal{U}}$ and $\mathrm{O}(\widetilde{\mathcal{U}}) = \emptyset$. We call $\widetilde{\mathcal{U}}$ an *extension of* $\mathcal{U}$, write $\mathcal{U} \to \widetilde{\mathcal{U}}$. We note that $\mathcal{U} \sim \widetilde{\mathcal{U}}$ if $\mathcal{U} \to \widetilde{\mathcal{U}}$.

Example 4.3. By Notation 3.1, let $\mathcal{U} \in \mathfrak{C}_9$ which is written by

$$213\check{2}31\check{2}13.$$

There exist four extensions of $\mathcal{U}$ as follows

$$213\check{2}31\check{2}13 \to_3 21\check{3}\check{1}\check{3}1\check{2}13 \to_2 2\check{1}\check{2}\check{1}\check{2}\check{1}\check{2}13,$$
$$213\check{2}31\check{2}13 \to_3 21\check{3}\check{1}\check{3}1\check{2}13 \to_6 21\check{3}\check{1}\check{3}\check{1}\check{3}\check{1}3,$$
$$213\check{2}31\check{2}13 \to_6 213\check{2}3\check{1}\check{3}\check{1}3 \to_3 21\check{3}\check{1}\check{3}\check{1}\check{3}\check{1}3,$$

and

$$213\check{2}31\check{2}13 \to_6 213\check{2}3\check{1}\check{3}\check{1}3 \to_5 213\check{2}\check{3}\check{2}\check{3}\check{2}\check{3}.$$

Definition 4.4. Let $\mathcal{U} \in \mathfrak{C}_p$. Set $\mathrm{K}(\mathcal{U}) = \{K = \{k, k+1, \dots, k+k'\} \subset \mathbb{Z}_p \setminus \bigcup \mathrm{O}(\mathcal{U})\colon\ (K+\delta) \cap \bigcup \mathrm{O}(\mathcal{U}) \neq \emptyset$ for $\delta = -1, 1\}$. We can write $\mathrm{K}(\mathcal{U}) = \{K_0, \dots, K_n\}$ such that $k_0 < \cdots < k_n$, where let $K_j = \{k_j, k_j + 1, \dots, k_j + k'_j\}$. We note $|\mathrm{O}(\mathcal{U})| = |\mathrm{K}(\mathcal{U})| = n + 1$ is even. Set $k_{\mathcal{U}} = \sum\{|K_j \setminus \mathrm{EC}(\mathcal{U})|\colon\ j$ is even$\}$, $k'_{\mathcal{U}} = \sum\{|K_j \setminus \mathrm{EC}(\mathcal{U})|\colon\ j$ is odd$\}$, and, $k(\mathcal{U}) = \min\{k_{\mathcal{U}}, k'_{\mathcal{U}}\}$.

Lemma 4.5. *Let $\mathcal{U} \in \mathfrak{C}_p$ and let $\widetilde{\mathcal{U}}$ be an extension of $\mathcal{U}$. Then $|\mathrm{EC}(\widetilde{\mathcal{U}})| = |\mathrm{EC}(\mathcal{U})| + 2k(\mathcal{U})$, where $k(\mathcal{U})$ is as in Definition 4.4.*

Proof. The proof is by induction on $k(\mathcal{U})$ that $|\mathrm{EC}(\widetilde{\mathcal{U}})| = |\mathrm{EC}(\mathcal{U})| + 2k(\mathcal{U})$. Let $n = |\mathrm{K}(\mathcal{U})| - 1$. We can write $\mathrm{O}(\mathcal{U}) = \{Z_0, \ldots, Z_n\}$ such that $s_j + t_j < k_j < k_j + k'_j < s_{j+1}$, where let $Z_j = \{s_j, k_j + 1, \ldots, s_j + t_j\}$ and let k_i and k'_j be as in Definition 4.4.

Suppose that $k(\mathcal{U}) = 1$. Then we have $n = 1$ and only two elementary extensions $\mathcal{U}'$, $\mathcal{U}''$ in $\mathcal{U}$ such that $\mathcal{U} \to_{s_0} \mathcal{U}'$, $\mathcal{U} \to_{s_1} \mathcal{U}''$, $\mathrm{O}(\mathcal{U}') = \mathrm{O}(\mathcal{U}'') = \emptyset$, and, $|\mathrm{EC}(\mathcal{U}')| = |\mathrm{EC}(\mathcal{U}'')| = |\mathrm{EC}(\mathcal{U})| + 2$.

Assuming the lemma holds for $k(\mathcal{U}) - 1$, we will prove it for $k(\mathcal{U}) > 1$. By Definition 4.2, we may assume that there exists an elementary extension $\mathcal{U}_1$ of $\mathcal{U}$ such that $\mathcal{U} \to_{s_1} \mathcal{U}_1 \to \widetilde{\mathcal{U}}$.

Case 1. $\min\{|K_0 \setminus \mathrm{EC}(\mathcal{U})|, |K_1 \setminus \mathrm{EC}(\mathcal{U})|\} > 1$, where let K_j be as in Definition 4.4. By Definitions 4.2 and 4.4, we have $|\mathrm{EC}(\mathcal{U}_1)| = |\mathrm{EC}(\mathcal{U})| + 2$ and $k(\mathcal{U}_1) = k(\mathcal{U}) - 1$. By induction, $|\mathrm{EC}(\widetilde{\mathcal{U}})| = |\mathrm{EC}(\mathcal{U}_1)| + 2k(\mathcal{U}_1) = |\mathrm{EC}(\mathcal{U})| + 2k(\mathcal{U})$.

Case 2. $|K_0 \setminus \mathrm{EC}(\mathcal{U})| = 1$ and $|K_1 \setminus \mathrm{EC}(\mathcal{U})| = 1$. By Definition 4.4, $|Z_0 \cup K_0 \cup Z_1 \cup K_1 \cup Z_2|$ is odd, we have

$$\mathrm{O}(\mathcal{U}_1) = \{Z_0 \cup K_0 \cup Z_1 \cup K_1 \cup Z_2, Z_3, Z_4, \ldots, Z_n\}.$$

Thus, we can write $\mathrm{K}(\mathcal{U}) = \{K_2, K_3, \ldots, K_n\}$. This shows that $k_{\mathcal{U}_1} = k_{\mathcal{U}} - 1$ and $k'_{\mathcal{U}_1} = k'_{\mathcal{U}} - 1$, therefore, $k(\mathcal{U}_1) = k(\mathcal{U}) - 1$. By induction, $|\mathrm{EC}(\widetilde{\mathcal{U}})| = |\mathrm{EC}(\mathcal{U}_1)| + 2k(\mathcal{U}_1) = |\mathrm{EC}(\mathcal{U})| + 2k(\mathcal{U})$.

Case 3. $|K_0 \setminus \mathrm{EC}(\mathcal{U})| = 1$ and $|K_1 \setminus \mathrm{EC}(\mathcal{U})| > 1$. By Definition 4.4, $|Z_0 \cup K_0 \cup Z_1 \cup (Z_1 + 1)|$ is even, thus, we have

$$\mathrm{O}(\mathcal{U}_1) = \{Z_2, \ldots, Z_n\}.$$

Thus, we can write $\mathrm{K}(\mathcal{U}) = \{K_2, K_3, \ldots, K'_n\}$, where let $K'_n = K_n \cup Z_0 \cup K_0 \cup Z_1 \cup K_1$. This shows that $k_{\mathcal{U}_1} = k_{\mathcal{U}} - 1$ and $k'_{\mathcal{U}_1} = k'_{\mathcal{U}} - 1$, therefore, $k(\mathcal{U}_1) = k(\mathcal{U}) - 1$. By induction, $|\mathrm{EC}(\widetilde{\mathcal{U}})| = |\mathrm{EC}(\mathcal{U}_1)| + 2k(\mathcal{U}_1) = |\mathrm{EC}(\mathcal{U})| + 2k(\mathcal{U})$.

Case 4. $|K_0 \setminus \mathrm{EC}(\mathcal{U})| > 1$ and $|K_1 \setminus \mathrm{EC}(\mathcal{U})| = 1$. It follows by the same method as in Case 3 that $|\mathrm{EC}(\widetilde{\mathcal{U}})| = |\mathrm{EC}(\mathcal{U}_1)| + 2k(\mathcal{U}_1) = |\mathrm{EC}(\mathcal{U})| + 2k(\mathcal{U})$, which completes the proof. □

Let $\mathcal{U} \in \mathfrak{C}_p$, and, let $\widetilde{\mathcal{U}}_0$ and $\widetilde{\mathcal{U}}_1$ be two extensions of $\mathcal{U}$. Then $|\mathrm{EC}(\widetilde{\mathcal{U}}_0)| = |\mathrm{EC}(\widetilde{\mathcal{U}}_1)|$ by Lemma 4.5.

Theorem 4.6. *Let $\mathcal{U}, \mathcal{V} \in \mathfrak{C}_p$ such that $\mathcal{U} \sim \mathcal{V}$ and $\mathrm{O}(\mathcal{V}) = \emptyset$. Then, $\ell([\mathcal{U}]) = |\mathrm{EC}(\mathcal{V})|$.*

Proof. First, we show the following:

Claim. *Let $\mathcal{U}_0, \mathcal{U}_1 \in \mathfrak{C}_p$ with $\mathcal{U}_0 \sim \mathcal{U}_1$, and, let $\widetilde{\mathcal{U}}_0$ and $\widetilde{\mathcal{U}}_1$ be two extensions of $\mathcal{U}_0$ and $\mathcal{U}_1$, respectively. Then $|\mathrm{EC}(\widetilde{\mathcal{U}}_0)| = |\mathrm{EC}(\widetilde{\mathcal{U}}_1)|$.*

We may assume that $\mathcal{U}_0 \sim_i \mathcal{U}_1$. In case of Lemma 3.4(2), it is easy to check that $k(\mathcal{U}_0) = k(\mathcal{U}_1)$, thus, $|\mathrm{EC}(\widetilde{\mathcal{U}}_0)| = |\mathrm{EC}(\widetilde{\mathcal{U}}_1)|$ by Lemma 4.5. In case of Lemma 3.4(1), we have $\mathcal{U}_0 \to \mathcal{U}_1$ or $\mathcal{U}_1 \to \mathcal{U}_0$. Since $\mathcal{U}_0 \to \widetilde{\mathcal{U}}_1$ or $\mathcal{U}_1 \to \widetilde{\mathcal{U}}_0$, we see $|\mathrm{EC}(\widetilde{\mathcal{U}}_0)| = |\mathrm{EC}(\widetilde{\mathcal{U}}_1)|$, which proves the claim.

Let $\mathcal{W} \in \mathfrak{C}_p$ such that $\ell([\mathcal{U}]) = |\mathrm{EC}(\mathcal{W})|$. By Lemma 3.7, we see that $\mathrm{O}(\mathcal{W}) = \emptyset$, thus, $\mathcal{W}$ is an extension of $\mathcal{W}$. Since $\mathcal{V}$ is an extension of $\mathcal{V}$ and $\mathcal{V} \sim \mathcal{W}$, by claim, we have $\ell([\mathcal{U}]) = |\mathrm{EC}(\mathcal{W})| = |\mathrm{EC}(\mathcal{V})|$, which proves the theorem. □

For $k = 0, 1$, let $\mathcal{U}_k \in \mathfrak{C}_p$ with $\mathrm{O}(\mathcal{U}_k) = \emptyset$. If $\mathcal{U}_0 \sim \mathcal{U}_1$, then $|\mathrm{EC}(\mathcal{U}_0)| = |\mathrm{EC}(\mathcal{U}_1)|$ by Theorem 4.6.

Theorem 4.7. *In Lemma 2.9, $p'_0 \ell([\mathcal{U}|_{\mathrm{Orb}(x_0, f)}]) = p'_1 \ell([\mathcal{U}|_{\mathrm{Orb}(x_1, f)}])$, where $p'_k = p/p_k$ for $k = 0, 1$.*

Proof. It suffices to show the following: Let $m, n \in \mathbb{N}$ with $m, n \geqslant 2$, $\mathcal{U} \in \mathfrak{C}_n$, and, $\gamma : \mathbb{Z}_{mn} \to \mathbb{Z}_n$ satisfying that $\gamma(i) = s_n^i(0) \in \mathbb{Z}_n$ for $i = 0, \ldots, mn - 1$.

$$\begin{tikzcd}
\mathbb{Z}_{mn} \arrow[r, "s_{mn}"] \arrow[d, "\gamma"'] & \mathbb{Z}_{mn} \arrow[d, "\gamma"] \\
\mathbb{Z}_n \arrow[r, "s_n"'] & \mathbb{Z}_n
\end{tikzcd}$$

Then $\ell([\gamma^{-1}(\mathcal{U})]) = m\ell([\mathcal{U}])$.

It is clear that $\gamma^{-1}(\mathcal{U}) \in \mathfrak{C}_{mn}$. Let $\mathcal{V} \in \mathfrak{C}_n$ such that $\ell([\mathcal{U}]) = |\mathrm{EC}(\mathcal{V})|$ and $\mathcal{U} \sim \mathcal{V}$. By definition we see that $\gamma^{-1}(\mathcal{U}) \sim \gamma^{-1}(\mathcal{V})$ and $\gamma^{-1}(\mathrm{EC}(\mathcal{V})) = \mathrm{EC}(\gamma^{-1}(\mathcal{V}))$. Since $\mathrm{O}(\gamma^{-1}(\mathcal{V})) = \emptyset$, by Theorem 4.6, we have $\ell([\gamma^{-1}(\mathcal{U})]) = |\mathrm{EC}(\gamma^{-1}(\mathcal{V}))| = m|\mathrm{EC}(\mathcal{V})| = m\ell([\mathcal{U}])$. □

Example 4.8. Let $k = 0, 1, \ldots$ and $i = 0, \ldots, k$.

(1) Let $\mathcal{U}_{6k+3,i} \in \mathfrak{C}_{6k+3}$, $\mathcal{U}_{6k+5,i} \in \mathfrak{C}_{6k+5}$, and, $\mathcal{U}_{6k+7,i} \in \mathfrak{C}_{6k+7}$ which are denoted by

$$\underbrace{\check{1}\check{2}\ldots\check{1}\check{2}}_{6i}\underbrace{132\ldots132}_{6(k-i)+3},$$

$$\underbrace{\check{1}\check{2}\ldots\check{1}\check{2}}_{6i+2}\underbrace{132\ldots132}_{6(k-i)+3},$$

and,

$$\underbrace{\check{1}\check{2}\ldots\check{1}\check{2}}_{6i+4}\underbrace{132\ldots132}_{6(k-i)+3},$$

respectively. Since $\mathrm{O}(\mathcal{U}_{6k+3,i}) = \emptyset$, $\mathrm{O}(\mathcal{U}_{6k+5,i}) = \emptyset$, and, $\mathrm{O}(\mathcal{U}_{6k+7,i}) = \emptyset$, by Theorem 4.6, we have $\ell([\mathcal{U}_{6k+3,i}]) = 6i$, $\ell([\mathcal{U}_{6k+5,i}]) = 6i + 2$, and $\ell([\mathcal{U}_{6k+7,i}]) = 6i + 4$. Thus, $\{\ell([\mathcal{U}])\colon \mathcal{U} \in \mathfrak{C}_{6k+3}\} \supset \{0, 6, \ldots, 6k\}$, $\{\ell([\mathcal{U}])\colon \mathcal{U} \in \mathfrak{C}_{6k+5}\} \supset \{2, 8, \ldots, 6k+2\}$, and, $\{\ell([\mathcal{U}])\colon \mathcal{U} \in \mathfrak{C}_{6k+7}\} \supset \{4, 10, \ldots, 6k+4\}$.

(2) Let $\mathcal{U}_{6k,i} \in \mathfrak{C}_{6k}$ $(k \geqslant 1)$, $\mathcal{U}_{6k+2,i} \in \mathfrak{C}_{6k+2}$, and, $\mathcal{U}_{6k+4,i} \in \mathfrak{C}_{6k+4}$, which are denoted by

$$\underbrace{\check{1}\check{2}\ldots\check{1}\check{2}}_{6i}\underbrace{132\ldots132}_{6(k-i)},$$

$$\underbrace{\check{1}\check{2}\ldots\check{1}\check{2}}_{6i+2}\underbrace{132\ldots132}_{6(k-i)},$$

and,

$$\underbrace{\check{1}\check{2}\ldots\check{1}\check{2}}_{6i+4}\underbrace{132\ldots132}_{6(k-i)},$$

respectively. Since $\mathrm{O}(\mathcal{U}_{6k,i}) = \emptyset$, $\mathrm{O}(\mathcal{U}_{6k+2,i}) = \emptyset$, and, $\mathrm{O}(\mathcal{U}_{6k+4,i}) = \emptyset$, by Theorem 4.6, we have $\ell([\mathcal{U}_{6k,i}]) = 6i$, $\ell([\mathcal{U}_{6k+2,i}]) = 6i + 2$, and $\ell([\mathcal{U}_{6k+4,i}]) = 6i + 4$. Thus, $\{\ell([\mathcal{U}])\colon \mathcal{U} \in \mathfrak{C}_{6k}\} \supset \{0, 6, \ldots, 6k\}$ $(k \geqslant 1)$, $\{\ell([\mathcal{U}])\colon \mathcal{U} \in \mathfrak{C}_{6k+2}\} \supset \{2, 8, \ldots, 6k+2\}$, and, $\{\ell([\mathcal{U}])\colon \mathcal{U} \in \mathfrak{C}_{6k+4}\} \supset \{4, 10, \ldots, 6k+4\}$.

Theorem 4.9. *Let $p \in \mathbb{N}$ with $p \geqslant 2$, let $q = -1, 0, 1$ with $p \equiv q \pmod 3$, and, let $r = 3|q| + q$. Then, $\{\ell([\mathcal{U}])\colon \mathcal{U} \in \mathfrak{C}_p\} = \{6i + r\colon 0 \leqslant 6i + r \leqslant p\ (i = 0, 1, \ldots)\}$.*

Proof. By Example 4.8, we see that $\{\ell([\mathcal{U}])\colon \mathcal{U} \in \mathfrak{C}_p\} \supset N = \{6i + r\colon\ 0 \leqslant 6i + r \leqslant p\ \ (i = 0, 1, \ldots)\}$. Let $\mathcal{U} = \{U_1, U_2, U_3\} \in \mathfrak{C}_p$ with $\ell([\mathcal{U}]) = |\mathrm{EC}(\mathcal{U})| = 2m$ for some $m = 0, 1, \ldots$. It suffices to show that $2m \in N$. By Lemma 3.7, we may assume that $\mathrm{EC}(\mathcal{U}) = \{0, \ldots, 2m - 1\}$. Moreover, we may assume that $0 \in U_1$ and $1 \in U_2$ if $m > 0$, and, that $0 \in U_1$, $1 \in U_3$, and, $2 \in U_2$ if $m = 0$. Since $2m \in U_1$ and $p - 1 \in U_2$, it is easy to check that

$$\mathcal{U} = \underbrace{1212\ldots12}_{2m}\underbrace{132132\ldots132}_{p-2m}.$$

By Example 4.8, there exists $j = 0, 1, \ldots$ such that $2m = 6i + r$. $\square$

5. Main results

Notation 5.1. Let $A = \{p_0, \ldots, p_k\}$ be a finite subset of $\mathbb{N}$ and let us write the least common multiple of A as $\mathrm{lcm}(A)$ or $\mathrm{lcm}(p_0, \ldots, p_k)$.

The following result is almost trivial, so the proof is left to the reader.

Lemma 5.2. *Let $k \in \mathbb{N}$ and $A = \{p_0, \ldots, p_k\}$ be a finite subset of $\mathbb{N}$. Then*

$$\mathrm{lcm}\left(\left\{\frac{p_i}{\gcd(p_i, p_j)}\colon i \neq j\right\}\right) = \frac{p_i}{\gcd(A)}$$

for each $i = 0, \ldots, k$.

Theorem 5.3. *Let $f : X \to X$ be a fixed-point free homeomorphism from a connected locally finite graph X into itself. If $\gcd(A) \in \{1, 3\}$ for some finite subset A of $\mathrm{Per}(f)$, then $\mathrm{col}(X, f) = 4$.*

Proof. Write $A = \{p_0, \ldots, p_k\}$. Let $\mathrm{Orb}(x_i, f)$ be a p_i-periodic orbit of x_i for f for each $i = 0, \ldots, k$. By Theorem 2.10, we may assume that $k \geqslant 1$ and $p_i \neq 3$ for all i. Suppose that there exists a coloring $\mathcal{U} = \{U_1, U_2, U_3\}$ of (X, f) such that $\mathcal{U}|_{\mathrm{Orb}(x_i, f)} \in \mathfrak{C}_{p_i}$ for each $i = 0, \ldots, k$.

Suppose that $\gcd(A) = 1$. By Theorem 4.7, we get $\frac{p_j}{\gcd(p_i, p_j)} \ell([\mathcal{U}|_{\mathrm{Orb}(x_i, f)}]) = \frac{p_i}{\gcd(p_i, p_j)} \ell([\mathcal{U}|_{\mathrm{Orb}(x_j, f)}])$ whenever $i \neq j$. Since X is path-connected, we have $\ell([\mathcal{U}|_{\mathrm{Orb}(x_i, f)}]) \neq 0$ for all i. Thus, $\frac{p_i}{\gcd(p_i, p_j)}$ and $\frac{p_j}{\gcd(p_i, p_j)}$ divide $\ell([\mathcal{U}|_{\mathrm{Orb}(x_i, f)}])$ and $\ell([\mathcal{U}|_{\mathrm{Orb}(x_j, f)}])$, respectively. Hence, for every $i = 0, \ldots, k$, $\frac{p_i}{\gcd(p_i, p_j)}$ divides $\ell([\mathcal{U}|_{\mathrm{Orb}(x_i, f)}])$ for all $j \neq i$. We show that $\ell([\mathcal{U}|_{\mathrm{Orb}(x_i, f)}]) = p_i$ for each i. Since $\gcd(A) = 1$, we see from Lemma 5.2 that $\mathrm{lcm}(\{\frac{p_i}{\gcd(p_i, p_j)}\colon\ i \neq j\}) = p_i$, hence, p_i divides $\ell([\mathcal{U}|_{\mathrm{Orb}(x_i, f)}])$. Since $\ell([\mathcal{U}|_{\mathrm{Orb}(x_i, f)}]) \leqslant p_i$ for each i, we have $\ell([\mathcal{U}|_{\mathrm{Orb}(x_i, f)}]) = p_i$ for each i. By Theorem 4.9, every p_i is even, a contradiction, which proves $\mathrm{col}(X, f) = 4$ when $\gcd(A) = 1$.

Suppose that $\gcd(A) = 3$. Since $p_i \neq 3$ for all i, we have $p_i \geqslant 6$ for all i. Let $m_0, n_0, \ldots, m_k, n_k \in \mathbb{N}$ with $\gcd(n_0, \ldots, n_k) = 1$ such that $p_i = 3n_i$, $p_i = 6m_i + 3$ or $p_i = 6m_i$ for each $i = 0, \ldots, k$. Thus, $m_i < n_i$ for each $i = 0, \ldots, k$. Since $\gcd(A) = 3$, we may assume that $p_0 = 6m_0 + 3$. By Theorem 4.7, $n_i \ell([\mathcal{U}|_{\mathrm{Orb}(x_0, f)}]) = n_0 \ell([\mathcal{U}|_{\mathrm{Orb}(x_i, f)}])$ for each $i = 0, \ldots, k$. By Theorem 4.9, for every $i = 0, \ldots, k$ there exists $\ell_i \in \mathbb{N}$ such that $\ell_i \leqslant m_i$ and $\ell([\mathcal{U}|_{\mathrm{Orb}(x_i, f)}]) = 6\ell_i$, and so $n_i = \frac{\ell_i n_0}{\ell_0}$ for all i. Since $\gcd(n_0, \ldots, n_k) = 1$, we get $\gcd(\ell_0 n_0, \ldots, \ell_k n_0) = \ell_0$, hence, n_0 divides ℓ_0. This contradicts the fact that $\ell_0 \leqslant m_0 < n_0$, which proves the theorem. $\square$

Notation 5.4. Let X be a finite graph and let $b(x, X)$ be a degree of a vertex x in X. Set $b_k(X) = |\{x \in X\colon\ b(x, X) = k\}|$ and $b(X) = \{b_k(X)\colon\ k \neq 2\} \setminus \{0\}$.

Corollary 5.5. *Let X be a connected finite graph. If $\gcd(b(X)) \in \{1, 3\}$, then $\mathrm{col}(X, f) = 4$ for all fixed-point free homeomorphism $f : X \to X$.*

Proof. Let $f : X \to X$ be a fixed-point free homeomorphism and $B_k(X) = \{x \in X\colon\ b(x, X) = k\}$ for each $k \in \mathbb{N} \setminus \{2\}$. We note that $f(B_k(X)) = B_k(X)$ for each $k \in \mathbb{N} \setminus \{2\}$. For every $k \in \mathbb{N} \setminus \{2\}$ there exist $x_{k,1}, \ldots, x_{k,n_k} \in B_k(X)$ such that $B_k(X) = \mathrm{Orb}(x_{k,1}, f) \cup \cdots \cup \mathrm{Orb}(x_{k,n_k}, f)$ and $\mathrm{Orb}(x_{k,i}, f) \cap \mathrm{Orb}(x_{k,j}, f) = \emptyset$ whenever $i \neq j$. Set $p_{k,i} = |\mathrm{Orb}(x_{k,i}, f)|$ for each $k \in \mathbb{N} \setminus \{2\}$ and each $i = 1, \ldots, n_k$. We show $g = \gcd(\{p_{k,i}\colon\ k \in \mathbb{N} \setminus \{2\},\ i = 1, \ldots, n_k\}) \in \{1, 3\}$. Since $b_k(X) = p_{k,1} + \cdots + p_{k,n_k}$, g divides $b_k(X)$ for all $k \in \mathbb{N} \setminus \{2\}$, and so g divides $\gcd(b(X))$. Therefore, $g \in \{1, 3\}$, and so $\mathrm{col}(X, f) = 4$ by Theorem 5.3, which proves the corollary. $\square$

Lemma 5.6. *Let $f : X \to X$ be a fixed-point free homeomorphism from a connected finite graph X into itself with $\mathrm{Per}(f) = \emptyset$. Then X is homeomorphic to the circle $\mathbb{S}^1$ and $\mathrm{col}(X, f) = 3$.*

Proof. It is clear that X is homeomorphic to the circle $\mathbb{S}^1$. Moreover, by [4], there exists a homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h \circ r_\theta = f \circ h$, where let r_θ be the θ-irrational rotation satisfying that $r_\theta(\cos 2\pi t, \sin 2\pi t) = (\cos 2\pi(t + \theta), \sin 2\pi(t + \theta))$ for $0 \leqslant t < 1$ and θ is an irrational number.

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{r_\theta} & \mathbb{S}^1 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{S}^1 & \xrightarrow[f]{} & \mathbb{S}^1 \end{array}$$

Hence, it suffices to show that $\mathrm{col}(\mathbb{S}^1, r_\theta) = 3$ for all irrational number θ with $0 < \theta < 1$.

Set $a_{1,i,0} = \frac{i-1}{3}$ for $i = 1, 2, 3$ and $U_{1,i} = \{(\cos 2\pi t, \sin 2\pi t)\colon\ t \in [a_{1,i,0} - \frac{1}{6}, a_{1,i,0} + \frac{1}{6}]\}$ for $i = 1, 2, 3$. It is clear that $\mathcal{U}_1 = \{U_{1,1}, U_{1,2}, U_{1,3}\}$ is a coloring for r_t for all $t \in (\frac{1}{2} - \frac{1}{6}, \frac{1}{2} + \frac{1}{6}) = (\frac{1}{3}, \frac{2}{3})$.

Let $n \in \mathbb{N}$ with $n \geqslant 2$. Define $a_{n,i,k} = \frac{i-1}{3n} + \frac{k}{n}$ for $i = 1, 2, 3$ and $k = 0, \ldots, n-1$ and $U_{n,i} = \{(\cos 2\pi t, \sin 2\pi t)\colon\ t \in [a_{n,i,k} - \frac{1}{6n}, a_{n,i,k} + \frac{1}{6n}]$ for some $k = 0, \ldots, n-1\}$ for $i = 1, 2, 3$. It is clear that $\mathcal{U}_n = \{U_{n,1}, U_{n,2}, U_{n,3}\}$ is a coloring for r_t for all $t \in (\frac{1}{2n} - \frac{1}{6n}, \frac{1}{2n} + \frac{1}{6n})$. Since $\frac{1}{2(n+1)} \in [\frac{1}{2n} - \frac{1}{6n}, \frac{1}{2n} + \frac{1}{6n})$, we have $(0, \frac{1}{3}) = \bigcup_{n=2}^{\infty} (\frac{1}{2n} - \frac{1}{6n}, \frac{1}{2n} + \frac{1}{6n})$, hence, $\mathrm{col}(\mathbb{S}^1, r_\theta) = 3$ for all $0 < \theta < \frac{1}{3}$. Similarly, $\mathrm{col}(\mathbb{S}^1, r_\theta) = 3$ for all $\frac{2}{3} < \theta < 1$, which proves the lemma. $\square$

Corollary 5.7. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a fixed-point free homeomorphism. Then, $f^3(x) = x$ for some $x \in X$ if and only if $\mathrm{col}(\mathbb{S}^1, f) = 4$.*

Lemma 5.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a fixed-point free homeomorphism from $\mathbb{R}$ into itself. Then $\mathrm{col}(\mathbb{R}, f) = 3$.*

Proof. For two points x, y in $\mathbb{R}$, denote the arc connecting between x and y by $[x, y]$ and set $(x, y) = [x, y] \setminus \{x, y\}$. Fix $x_0 \in \mathbb{R}$. There exist $x_1, x_2 \in (x_0, f(x_0))$ such that $x_1 \in (x_0, x_2)$. Since f is a fixed-point free homeomorphism, we have that $f^{-1}(x_1), f^{-1}(x_2) \in (f^{-1}(x_0), x_0)$ and $f^k([f^{-1}(x_0), x_0]) = [f^{k-1}(x_0), f^k(x_0)]$ for each integer k. Set $U_1 = \bigcup_{n\in\mathbb{Z}} f^{2n}([f^{-1}(x_2), x_1])$, $U_2 = \bigcup_{n\in\mathbb{Z}} f^{2n}([x_1, f(x_0)])$, and, $U_3 = \bigcup_{n\in\mathbb{Z}} f^{2n}([f(x_0), f(x_2)])$. It is clear that $\mathcal{U} = \{U_1, U_2, U_3\}$ is a coloring for (X, f), which proves the lemma. $\square$

Theorem 5.9. *Let $f : X \to X$ be a fixed-point free homeomorphism from a connected locally finite graph X into itself with* $\mathrm{Per}(f) = \emptyset$. *Then* $\mathrm{col}(X, f) = 3$.

Proof. By Lemmas 5.6 and 5.8, we may assume that X is an infinite graph with vertices. We show that there exist a closed connected subgraph F of X and two closed subsets F_0 and F_1 of X such that $F_0 \cup F_1$ is the boundary of F in X, $F_0 \cap F_1 = \emptyset$, $f(F_0) = F_1$, and, $X = \bigcup_{n\in\mathbb{Z}} f^n(F)$.

There exists a connected subgraph C of X such that $f^n(x) \neq y$ for any two vertices x, y in C and any $n \in \mathbb{Z}$, and, for every edge e of X, $|e \cap C^{(0)}| = 2$ implies $e \subset C$, where $C^{(0)}$ is the set of all vertices of C. There exists a maximal connected subgraph E in C's satisfying the conditions above. Set $\overline{E} = \bigcup_{n\in\mathbb{Z}} f^n(E)$ containing $X^{(0)}$. Assume that we have an edge $e \not\subset \overline{E}$ with $e \cap E \neq \emptyset$ and $f(e) \cap E = \emptyset$. Let $e^{(0)} = \{v, w\}$ with $v \in E \not\ni w$, $w' = f^{-1}(w)$ and $e' = f^{-1}(e)$ which is an edge of X. By the maximum of E, we have $w' \in E$ and $e' \not\subset \overline{E}$. There exist half-edges e_h and e'_h of e and e', respectively, such that $v \in e_h$, $w' \in e'_h$, $e_h \cap f(e'_h)$ is one point, and, $e = e_h \cup f(e'_h)$. Set

$$F = E \cup \bigcup \bigl\{e_h \cup e'_h\colon\ e \not\subset \overline{E},\ e \cap E \neq \emptyset,\ f(e) \cap E = \emptyset\bigr\}.$$

We check at once that F has the conditions above.

There exists a surjective map $g : F \to \mathbb{I}$ such that $g^{-1}(i) = F_i$ for $i = 0, 1$. Thus, there exists a surjective map $h : X \to \mathbb{R}$ with $h|_F = g$ and a fixed-point free homeomorphism $f' : \mathbb{R} \to \mathbb{R}$ such that $f^n(F_0) = h^{-1}(n)$ for all $n \in \mathbb{Z}$ and $h \circ f = f' \circ h$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ \mathbb{R} \arrow[r, "f'"'] & \mathbb{R} \end{tikzcd}$$

By Lemma 5.8, we have a 3-coloring $\mathcal{U} = \{U_1, U_2, U_3\}$ for $(\mathbb{R}, f')$, therefore, we have a 3-coloring $\{h^{-1}(U_1), h^{-1}(U_2), h^{-1}(U_3)\}$ for (X, f), which proves the theorem. $\square$

Example 5.10. Let G be an infinite group with a finite generating set S and let X be the Cayley graph of G with respect to S. For every $g \in G$ define a homeomorphism $g : X \to X$ by $g(h) = gh$ for each $h \in G \subset X$ (see [7]). If $\{g^n\colon\ n \in \mathbb{N}\}$ is infinite, $\mathrm{Per}(g) = \emptyset$, thus, $\mathrm{col}(X, g) = 3$ by Theorem 5.9.

Theorem 5.11. *Let X be a connected locally finite graph and let $f : X \to X$ be a fixed-point free homeomorphism with* $\mathrm{Per}(f) \neq \emptyset$. *If* $\gcd(\mathrm{Per}(f)) \notin \{1, 3\}$, *then* $\mathrm{col}(X, f) = 3$.

Proof. By [1], we may assume that X is an infinite graph. Set $B(X) = \{x \in X\colon\ b(x, X) \geqslant 3\} \neq \emptyset$ (see Notation 5.4) and $\mathrm{P}(f) = \{x \in X\colon\ f^{|\mathrm{Orb}(x,f)|}(x) = x\}$. Fix $x_0 \in \mathrm{P}(f)$. By Lemma 2.9, there exist $n \in \mathbb{N}$ and an arc A connecting between x_0 and $f(x_0)$ in X such that $|\mathrm{Orb}(x_0, f)|$ divides n and $A = f^n(A)$. Set $Y = \bigcup_{i=0}^{n-1} f^i(A)$ which is a connected finite graph satisfying $f(Y) = Y$. Since X is a connected infinite graph, we see that $Y \cap B(X) \neq \emptyset$, therefore, $\mathrm{P}(f) \cap B(X) \neq \emptyset$ because $Y \cap B(X) \subset \mathrm{P}(f)$. It is clear that $f(\mathrm{P}(f) \cap B(X)) = \mathrm{P}(f) \cap B(X)$.

There exists a nonempty set $P \subset \mathrm{P}(f) \cap B(X)$ such that $\mathrm{P}(f) \cap B(X) = \bigcup_{x\in P} \mathrm{Orb}(x, f)$, and, $\mathrm{Orb}(x, f) \cap \mathrm{Orb}(x', f) = \emptyset$ whenever $x \neq x'$ for $x, x' \in P$. If $\gcd(\mathrm{Per}(f))$ is even, set $U_1 = \{f^{2i}(x)\colon\ x \in P,\ i \in \mathbb{N}\}$, $U_2 = \{f^{2i-1}(x)\colon\ x \in P,\ i \in \mathbb{N}\}$, and, $U_3 = \emptyset$. If $g = \gcd(\mathrm{Per}(f))$ is odd, set $U_1 = \{f^{gj+2i}(x)\colon\ x \in P,\ i, j \in \mathbb{N}$, and, $1 \leqslant 2i < g\}$, $U_2 = \{f^{gj+2i-1}(x)\colon\ x \in P,\ i, j \in \mathbb{N}$, and, $1 \leqslant 2i - 1 < g\}$, and, $U_3 = \{f^{gj}(x)\colon\ x \in P,\ j \in \mathbb{N}\}$. Thus, $\mathcal{U} = \{U_1, U_2, U_3\}$ is a coloring of $(\mathrm{P}(f) \cap B(X), f|_{\mathrm{P}(f)\cap B(X)})$.

Set $\mathcal{C} = \{C\colon\ C$ is the closure of a component of $X \setminus \mathrm{P}(f) \cap B(X)\}$. Since X is a locally finite graph, for each $C \in \mathcal{C}$ there exists $k \in \mathbb{N}$ such that $f^k(C) = C$. Set $\widetilde{C} = \bigcup_{i\in\mathbb{N}} f^i(C)$ for each $C \in \mathcal{C}$. There exists $\mathcal{C}_0 \subset \mathcal{C}$ such that $X = \bigcup_{C\in\mathcal{C}_0} \widetilde{C}$ and, for $C, C' \in \mathcal{C}_0$, $|\widetilde{C} \cap \widetilde{C'}|$ is finite if and only if $C \neq C'$. Set $\mathcal{C}_1 = \{C \in \mathcal{C}_0\colon\ |C \cap \mathrm{P}(f) \cap B(X)| = 2\}$. We note that C is homeomorphic to $[0, 1]$ for each $C \in \mathcal{C}_1$. As the proof of [1, Proposition 3.10], for every $C \in \mathcal{C}_1$ there exists a 3-coloring $\mathcal{U}_C = \{U_{C,1}, U_{C,2}, U_{C,3}\}$ of $(\widetilde{C}, f|_{\widetilde{C}})$ such that $\mathcal{U}_C|\widetilde{C} \cap \mathrm{P}(f) \cap B(X) = \mathcal{U}|\widetilde{C} \cap \mathrm{P}(f) \cap B(X)$. We note that $|\{i\colon\ C \cap U_i \neq \emptyset\}| = 1$ for each $C \in \mathcal{C}_0 \setminus \mathcal{C}_1$. Set $\widetilde{U}_i = \bigcup_{C\in\mathcal{C}_1} U_{C,i} \cup \bigcup\{C \in \mathcal{C}_0 \setminus \mathcal{C}_1\colon\ C \cap U_i \neq \emptyset\}$ for each $i = 1, 2, 3$. By construction, $\{\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3\}$ is a 3-coloring of (X, f), which proves the theorem. $\square$

Example 5.12. We cannot omit locally finite in Theorem 1.5.

Set $A_\delta = \{6n + 2\delta\colon\ n$ is an integer$\} \subset \mathbb{R}$ for each $\delta = 0, 1, 2$ and $X = \mathbb{R}/\{A_0, A_1, A_2\}$. It is clear that X is a connected graph, but not locally finite. A homeomorphism $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = t + 2$ for each $t \in \mathbb{R}$ induces the homeomorphism

$f : X \to X$ with 3-periodic point and $h \circ g = f \circ h$, where $h : \mathbb{R} \to X$ is the quotient map. Set $a_n = 2n + 2^{-n}$ if n is a nonnegative integer, $a_n = 2(n+1) - 2^n$ if n is a negative integer, and, $U_\delta = \bigcup\{[a_{3k+\delta-1}, a_{3k+\delta}] \subset \mathbb{R}\colon k \text{ is an integer}\}$ for each $\delta = 0, 1, 2$. It is easy to check that $\{U_1, U_2, U_3\}$ is a 3-coloring for $(\mathbb{R}, g)$ and $\{h(U_1), h(U_2), h(U_3)\}$ is a 3-coloring for (X, f), therefore, $\operatorname{col}(X, f) = 3$.

In [1, Example 3.15], it is shown that $\operatorname{col}(\mathbb{Z}_4 * \mathbb{Z}_4, f) = 3$ for all fixed-point free homeomorphism $f : \mathbb{Z}_4 * \mathbb{Z}_4 \to \mathbb{Z}_4 * \mathbb{Z}_4$. Set $F_{p,q} = \{f : \mathbb{Z}_p * \mathbb{Z}_q \to \mathbb{Z}_p * \mathbb{Z}_q\colon f \text{ is a fixed-point free homeomorphism}\}$.

Corollary 5.13. *Let $A_0 = \{(2, 4), (4, 4)\}$, $A_1 = \{(p, q) \in \mathbb{N} \times \mathbb{N}\colon p, q \geqslant 2$ and $\gcd(p, q) \in \{1, 3\}\}$, and, $A_2 = \{(p, q) \in \mathbb{N} \times \mathbb{N}\colon p, q \geqslant 2$ and $(p, q) \notin A_0 \cup A_1\}$. Then,*

(1) *$(p, q) \in A_0$ if and only if $\{\operatorname{col}(\mathbb{Z}_p * \mathbb{Z}_q, f)\colon f \in F_{p,q}\} = \{3\}$,*
(2) *$(p, q) \in A_1$ if and only if $\{\operatorname{col}(\mathbb{Z}_p * \mathbb{Z}_q, f)\colon f \in F_{p,q}\} = \{4\}$, and,*
(3) *$(p, q) \in A_2$ if and only if $\{\operatorname{col}(\mathbb{Z}_p * \mathbb{Z}_q, f)\colon f \in F_{p,q}\} = \{3, 4\}$.*

Proof. It follows from [1, Example 3.15] that $\operatorname{col}(\mathbb{Z}_4 * \mathbb{Z}_4, f) = 3$ for all $f \in F_{4,4}$. Let $f \in F_{2,4}$. It is clear that $f(\mathbb{Z}_2) = \mathbb{Z}_2$. Suppose that there exist $z_{4,j} \in \mathbb{Z}_4$ and $n \in \mathbb{N}$ such that $f^n(z_{2,0} * z_{4,j} * z_{2,1}) = z_{2,0} * z_{4,j} * z_{2,1}$. Thus, if n is odd, then there exists $z \in z_{2,0} * z_{q,j} * z_{2,1} \setminus \{z_{2,0}, z_{2,1}\}$ such that $f^n(z) = z$. If $n = 3$, there exists $z_{4,j'} \in \mathbb{Z}_4$ such that $f(z_{2,0} * z_{4,j'} * z_{2,1}) = z_{2,0} * z_{4,j'} * z_{2,1}$, and so f has a fixed point in $z_{2,0} * z_{4,j'} * z_{2,1} \setminus \{z_{2,0}, z_{2,1}\}$, a contradiction. Therefore, $f^2(z_{2,0} * z_{4,j} * z_{2,1}) = z_{2,0} * z_{4,j} * z_{2,1} \neq f(z_{2,0} * z_{4,j} * z_{2,1})$ for each $z_{4,j} \in \mathbb{Z}_4$. This shows that $\gcd(\operatorname{Per}(f)) = 2$, and so $\operatorname{col}(\mathbb{Z}_2 * \mathbb{Z}_4, f) = 3$ by Theorem 1.5.

Let $f \in F_{2,q}$ and q an odd number, i.e., $(2, q) \in A_1$. It is clear that $f(\mathbb{Z}_2) = \mathbb{Z}_2$. There exist $z_{q,j} \in \mathbb{Z}_q$ and an odd number $n \in \mathbb{N}$ such that $f^n(z_{2,0} * z_{q,j} * z_{2,1}) = z_{2,0} * z_{q,j} * z_{2,1}$, thus, there exists $z \in z_{2,0} * z_{q,j} * z_{2,1} \setminus \{z_{2,0}, z_{2,1}\}$ such that $f^n(z) = z$. Since $\gcd(\operatorname{Per}(f)) = 1$, $\operatorname{col}(\mathbb{Z}_2 * \mathbb{Z}_q, f) = 4$ by Theorem 1.5.

Let $(p, q) = (3, 3)$ and $f \in F_{p,q}$. It is clear that $f(\mathbb{Z}_p \cup \mathbb{Z}_q) = \mathbb{Z}_p \cup \mathbb{Z}_q$. As in the proof of [1, Example 3.15(2)], we have that $\mathbb{Z}_p = \{z_{3,0}, f^2(z_{3,0}), f^4(z_{3,0})\}$ and $\mathbb{Z}_q = \{f(z_{3,0}), f^3(z_{3,0}), f^5(z_{3,0})\}$, or that $f(\mathbb{Z}_p) = \mathbb{Z}_p$ and $f(\mathbb{Z}_q) = \mathbb{Z}_q$. Since $f^3(z_{3,0} * f(z_{3,0})) = z_{3,0} * f(z_{3,0})$, there exists $z \in z_{3,0} * f(z_{3,0})$ such that $f^3(z) = z$. It follows from Theorem 1.5 that $\operatorname{col}(\mathbb{Z}_p * \mathbb{Z}_q, f) = 4$.

Let $(p, q) \in A_1 \setminus \{(3, 3)\}$ with $p, q \geqslant 3$ and $f \in F_{p,q}$. Since $f(\mathbb{Z}_p) = \mathbb{Z}_p$ and $f(\mathbb{Z}_q) = \mathbb{Z}_q$, $\operatorname{col}(\mathbb{Z}_p * \mathbb{Z}_q, f) = 4$ by Corollary 5.5.

If $\mathbb{Z}_2 * \mathbb{Z}_2$ is homeomorphism to $\mathbb{S}^1$, $\{\operatorname{col}(\mathbb{Z}_2 * \mathbb{Z}_2, f)\colon f \in F_{2,2}\} = \{3, 4\}$.

Let $(p, q) \in A_2$ and $s_{p,q} : \mathbb{Z}_p * \mathbb{Z}_q \to \mathbb{Z}_p * \mathbb{Z}_q \in F_{p,q}$ as in Example 2.8. Since $\gcd(p, q) \neq 1, 3$, by Example 2.8, we have $\operatorname{col}(\mathbb{Z}_p * \mathbb{Z}_q, s_{p,q}) = 3$.

Let $(p, q) \in A_2$ and $f \in F_{p,q}$ such that $p \geqslant 2$ and $q \geqslant 6$ are even. Since $\mathbb{Z}_q$ is homeomorphic to the disjoint sum $\mathbb{Z}_3 \cup \mathbb{Z}_{q-3}$ of $\mathbb{Z}_3$ and $\mathbb{Z}_{q-3}$, we have $s = s_{p,3} \cup s_{p,q-3} : \mathbb{Z}_p * (\mathbb{Z}_3 \cup \mathbb{Z}_{q-3}) \to \mathbb{Z}_p * (\mathbb{Z}_3 \cup \mathbb{Z}_{q-3}) \in F_{p,q}$, where let $s_{p,3} : \mathbb{Z}_p * \mathbb{Z}_3 \to \mathbb{Z}_p * \mathbb{Z}_3$ and $s_{p,q-3} : \mathbb{Z}_p * \mathbb{Z}_{q-3} \to \mathbb{Z}_p * \mathbb{Z}_{q-3}$ be as in Example 2.8. Since p is even, $\gcd(\operatorname{Per}(f)) = 1$, and so $\operatorname{col}(\mathbb{Z}_p * \mathbb{Z}_q, s) = 4$ by Theorem 1.5.

Let $(p, q) \in A_2$ and $f \in F_{p,q}$ such that $p \geqslant 5$ and $q \geqslant 5$ are odd. Since $\mathbb{Z}_q$ is homeomorphic to the disjoint sum $\mathbb{Z}_2 \cup \mathbb{Z}_{q-2}$ of $\mathbb{Z}_2$ and $\mathbb{Z}_{q-2}$, we have $s' = s_{p,2} \cup s_{p,q-2} : \mathbb{Z}_p * (\mathbb{Z}_2 \cup \mathbb{Z}_{q-2}) \to \mathbb{Z}_p * (\mathbb{Z}_2 \cup \mathbb{Z}_{q-2}) \in F_{p,q}$, where let $s_{p,2} : \mathbb{Z}_p * \mathbb{Z}_2 \to \mathbb{Z}_p * \mathbb{Z}_2$ and $s_{p,q-2} : \mathbb{Z}_p * \mathbb{Z}_{q-2} \to \mathbb{Z}_p * \mathbb{Z}_{q-2}$ be as in Example 2.8. Since p is odd, $\gcd(\operatorname{Per}(f)) = 1$, and so $\operatorname{col}(\mathbb{Z}_p * \mathbb{Z}_q, s') = 4$ by Theorem 1.5. □

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